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Impedances of an Elliptic Waveguide (For the $_eH_1$ Mode)*

G. R. VALENZUELA†, MEMBER, IRE

Summary—**The power-voltage, power-current and voltage-current impedances for the elliptical waveguide for the fundamental mode ($_eH_1$ mode) are obtained by two different methods.**

The first method consists of using the exact fields inside a perfectly conducting elliptical pipe. Numerical results were obtained by numerical integration of the integrals involving Mathieu functions by the Gaussian Quadratures method by a digital computer.

In the second method approximate fields which satisfy the boundary conditions were used. By this approximate method, actual expressions for the impedances are obtained as a function of minor to major diameter ratio with no need of numerical integration.

The actual expressions for the impedance obtained by the approximate method give the impedance for elliptical waveguide within six per cent. On the basis of comparison with the exact numerical solution the expressions for the approximate impedance give the impedance of elliptical waveguide within three per cent if they are scaled by 1.03.

Introduction

CHU[1] in 1938 obtained numerical results for the exact cutoff wavelength for several modes in elliptical waveguides. He also obtained numerical results for the attenuation in elliptical waveguide.

Kihara[2] published a paper in 1947 using the variational method to determine the propagation constant of hollow pipes and cavities. Kihara was able to obtain the propagation constant of an elliptical waveguide within one per cent, in the first approximation, using trial fields.

In 1958 Harrowell[3] obtained the impedance of elliptical waveguide by using an approximate method. He showed that the magnetic field lines inside a circular waveguide were approximately ellipses. Therefore, he was able to introduce conducting ellipses without disturbing the fields. Harrowell did not mention within what degree of accuracy his impedances would compare with the exact value.

Harrowell's voltage-current impedance agrees perfectly with our exact impedance, but his impedances involving power differ from our exact values; this difference is greater for eccentricities close to unity.[4]

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† The Johns Hopkins Radiation Lab., Baltimore, Md.

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[4] In private correspondence we pointed out to R. V. Harrowell that $J_1'(kr)=0$ was not correct inside the circular waveguide. Harrowell acknowledged this. Despite the fact that he modified his impedances, a discrepancy still exists.

In the approximate method used in this paper Kihara's trial fields are used to obtain the expression for the impedances.

The Impedance in Waveguide

The impedances in a waveguide are arbitrarily defined as,[5]

$$Z_{WV} = \frac{VV^*}{2W}, \qquad Z_{WI} = \frac{2W}{II^*}, \qquad Z_{VI} = \frac{V}{I}$$

for sinusoidal fields, where * as superscript indicates the complex conjugate. V is the maximum voltage across the waveguide, I is the total axial current and W is the power flowing down the waveguide.

The impedances obtained by these expressions are all different.

The Exact Impedance

The exact impedance is obtained by using the exact fields in the elliptical pipe. The fields components of an H wave in an orthonormal system u_1, u_2, u_3 of coordinates can be expressed as[6]

$$E_1 = \frac{-\mu}{h_2}\frac{\partial^2 \Pi_3^*}{\partial t \partial u_2}, \quad E_2 = \frac{\mu}{h_1}\frac{\partial^2 \Pi_3^*}{\partial t \partial u_1}, \quad E_3 = 0,$$

$$H_1 = \frac{1}{h_1}\frac{\partial^2 \Pi_3^*}{\partial t \partial u_1}, \quad H_2 = \frac{1}{h_2}\frac{\partial^2 \Pi_3^*}{\partial z \partial u_2}, \quad H_3 = (k^2 - \beta^2)\Pi_3^*,$$

where

Π_3^* = the magnetic Hertz vector in the direction u_3,
μ = permeability of the medium,
k = free space propagation constant,

$$\omega\sqrt{\mu_0 \epsilon_0} = 2\pi/\lambda_0,$$

β = waveguide propagation constant $2\pi/\lambda_g$,

and h_1 and h_2 are the metric coefficients of the respective coordinates.

In elliptical coordinates,

$$x = q \cosh u \cos v,$$

$$y = q \sinh u \sin v, \text{ and}$$

$$z = z$$

where q is the semifocal distance; u and v are the radial and angular variables, respectively.

The magnetic Hertz vector must satisfy the wave equation. In elliptical boundaries the magnetic Hertz vector satisfies the wave equation if it is expressed as a function of Mathieu functions. For the fundamental mode (${}_eH_1$ mode),

$$\Pi_3^* = \frac{B}{(k^2 - \beta^2)} Je_1(u) Se_1(v)$$

where

$$Se_1(v) = \sum_{k=0}^{\infty} De^{(1)}_{(2k+1)} \cos (2k + 1)v,$$

$$Je_1(u) = \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} De^{(1)}_{(2k+1)} J_{2k+1}(h \cosh u)$$

and the coefficients $De^{(1)}_{(2k+1)}$ can be found in National Bureau of Standards' tables.[7]

B is a constant, $Se_1(v)$ is an even Matheiu function, $Je_1(u)$ is an even modified Matheiu function of the first kind, and the Mathieu functions have $h = q(k^2 - \beta^2)^{1/2}$ as a parameter.

Assuming an $e^{i(\omega t - \beta z)}$ dependence, the field components in elliptical coordinates are

$$E_u = -B\frac{i\omega\mu}{q_1(k^2 - \beta^2)} Je_1(u) Se_1'(v),$$

$$E_v = B\frac{i\omega\mu}{q_1(k^2 - \beta^2)} Je_1'(u) Se_1(v),$$

$$E_z = 0,$$

$$H_u = -B\frac{i\beta}{q_1(k^2 - \beta^2)} Je_1'(u) Se_1(v),$$

$$H_v = -B\frac{i\beta}{q_1(k^2 - \beta^2)} Je_1(u) Se_1'(v),$$

and

$$H_z = BJe_1(u)Se_1(v)$$

where

$$q_1 = q(\sinh^2 u + \sin^2 v)^{1/2}$$

and the prime denotes the derivative with respect to u or v.

For readers who desire to know more about Mathieu functions and their derivatives, papers by Wiltse and King[8,9] are very appropriate.

A. The Voltage

Schelkunoff[10] showed that the maximum voltage across a circular waveguide can be obtained from the longitudinal component of the magnetic field.

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[9] M. J. King and J. C. Wiltse, "Derivatives, Zeros, and Other Data Pertaining to Mathieu Functions," The Johns Hopkins Rad. Lab., Baltimore, Md., Tech. Rept. AF-57; December, 1958.

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In elliptical coordinates, the maximum voltage across the waveguide can also be obtained from the longitudinal magnetic field:

$$i\omega\mu H_z(u,v) = i\omega\mu(k^2-\beta^2)\Pi_z^*$$
$$= \frac{i\omega\mu}{q_1^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial\Pi_z^*}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{\partial\Pi_z^*}{\partial v}\right)\right]$$

substituting

$$\frac{\partial\Pi_z^*}{\partial v} = \frac{q_1E_u}{-i\omega\mu} \quad \text{and} \quad \frac{\partial\Pi_z^*}{\partial u} = \frac{q_1E_v}{i\omega\mu}$$

into H_z, and then integrating, it can be easily shown that

$$V = i\omega\mu\int_0^{u_0}\int_{-\pi/2}^{\pi/2} H_z(u,v)ds_1ds_2 = 2\int_0^{u_0} E_u\left(u_1,\frac{\pi}{2}\right)ds_1.$$

Accordingly,

$$V = i\omega\mu\int_0^{u_0}\int_{-\pi/2}^{\pi/2} H_z ds_1ds_2$$
$$= i\omega\mu Bq^2\int_0^{u_0}\int_{-\pi/2}^{\pi/2}(\sinh^2 u + \sin^2 v)Je_1(u)Se_1(v)dudvds_1ds_2$$

$ds_1ds_2 = q_1^2dudv$ (differential area).

If we let I_1 represent the integral, then the final expression for the maximum voltage across the waveguide is

$$V = Bi\omega\mu q^2 I_1. \qquad (1)$$

B. The Current

If we consider that the elliptical waveguide is in a position such that the major diameter is horizontal, the axial current flow is in one direction on the top face and in the opposite direction in the bottom. Then the total axial current can be obtained by the integral of the transverse tangential magnetic field on the upper half of the elliptical pipe:

$$I = \int_0^{\pi} H_v(u_0,v)ds_2 = \frac{-Bi\beta}{(k^2-\beta^2)}Jc_1(u_0)\int_0^{\pi} Se_1'(v)dv$$

and the final expression for the current is

$$I = B\frac{2i\beta}{k_0^2}Je_1(u_0) \qquad (2)$$

where $k_0^2=(k^2-\beta^2)$.

C. The Power

The power flowing down the waveguide is

$$W = \frac{1}{2}\int_{\text{volume}} E\times H^*d\tau$$
$$= \frac{1}{2}\int_0^{u_0}\int_0^{2\pi}(E_uH_v^* - E_vH_v^*)ds_1ds_2\cdot 1$$

and

$$W = |B|^2\frac{\omega\mu\beta}{2k_0^4}\int_0^{u_0}\int_0^{2\pi}[Je_1^2(u)Se_1'(v) + Je_1'^2(u)Se_1^2(v)]dudv.$$

Although

$$\int_0^{2\pi} Se_1^2(v)dv = \pi\sum_{k=0}^{\infty}\left[De_{(2k+1)}^{(1)}\right]^2$$

and

$$\int_0^{2\pi} Se_1'^2(v)dv = \pi\sum_{k=0}^{\infty}(2k+1)^2\left[De_{(2k+1)}^{(1)}\right]^2.$$

In the paper the values of the Mathieu functions were obtained directly by integrating the Mathieu equation, these expressions were not used, and the actual integration was performed.

If we let the integral be represented by I_2, the power is

$$W = |B|^2\frac{\omega\mu\beta}{2k_0^4}I_2. \qquad (3)$$

Now the impedances can be easily obtained. Let $Z_0=120\pi(\lambda_g/\lambda)$ and $h=q(k^2-\beta^2)^{1/2}$, then

$$Z_{WI} = \frac{h^4I_1^2}{I_2}\cdot Z_0$$

and

$$Z_{WI} = \frac{I_2}{4Je_1^2(u_0)}\cdot Z_0. \qquad (4)$$

At the cut off $\beta=0$

$$h = q\frac{2\pi}{\lambda_c}$$

and the cut off wavelength is

$$\lambda_c = \frac{2\pi q}{h} = \frac{\pi e A}{h}$$

where

e = eccentricity and
A = major diameter of the ellipse.

Numerical Results

The values of the Mathieu functions and their derivatives required for the numerical integrations were obtained directly by integrating the Mathieu differential equations by the Univac-1103A computer.

The characteristic values be_1 were obtained from the expression $be_1=a+s/2$, where $s=h^2$ and a is a continued fraction expansion of s. a was evaluated by using a routine available for the IBM 650 computer for each s. For more details see a paper by Valenzuela and Bitterli.[11]

[11] G. R. Valenzuela and C. V. Bitterli, "Tables of Even and Even Modified Mathieu Functions of Order One of the First Kind and Their Derivatives," The Johns Hopkins University Appl. Phys. Lab., Silver Spring, Md., Rept. No. APL-CM-966; November, 1959.

TABLE I

B/A	e = eccentricity	$S=h^2$	u_0	I_1	I_2	$Je_1(u_0)$
1.0000	0.0	0.0	∞	∞	2.000	0.7300
0.92367	0.3821	0.50	1.6135400	3.8394620	1.9616937	0.7420
0.8415	0.540	1.00	1.2263444	1.8158124	1.9072496	0.7560
0.7517	0.660	1.50	0.97686012	1.1244031	1.8232323	0.7720
0.6512	0.759	2.00	0.77738676	0.76092589	1.6947832	0.7890
0.5348	0.845	2.50	0.59694283	0.52235541	1.4991724	0.8067
0.3862	0.924	3.00	0.40732370	0.32811390	1.1679826	0.8255
0.1181	0.993	3.50	0.11873645	0.089946928	0.38690746	0.8455
0.0	1.000	∞	0.0	0.0	0.0	∞

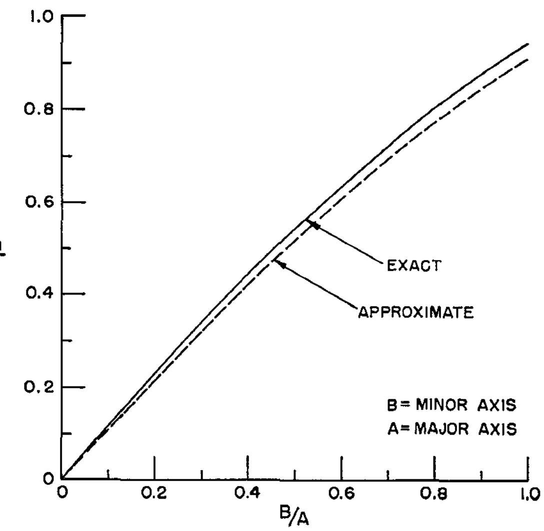

Fig. 1—Power-current impedance vs minor-to-major diameter ratio.

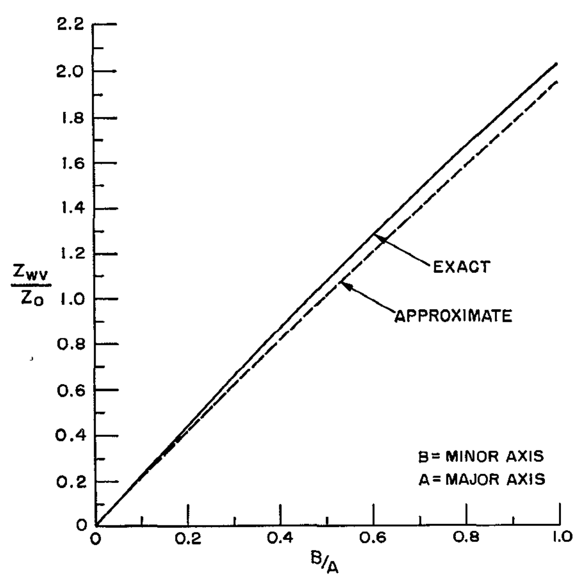

Fig. 2—Power-voltage impedance vs minor-to-major diameter ratio.

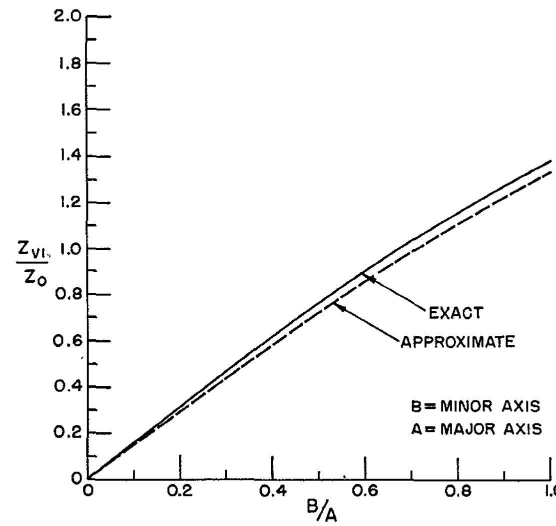

Fig. 3—Voltage-current impedance vs minor-to-major diameter ratio.

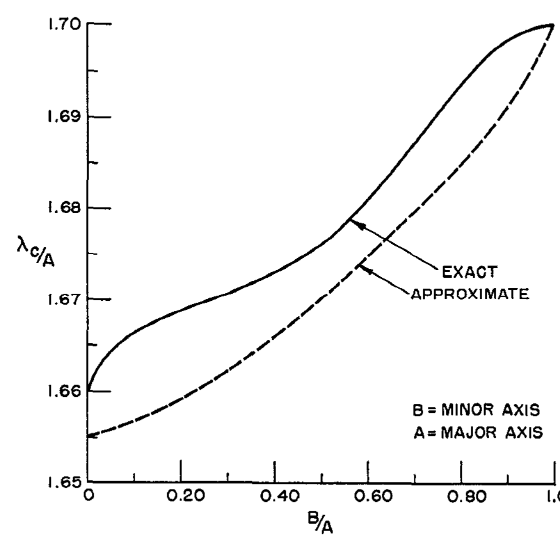

Fig. 4—Cutoff wavelength per major diameter vs minor-to-major axis ratio.

The integration of the Mathieu functions were performed using the Gaussian Quadratures method, sixteen points. The lowest root of the radial Mathieu functions were obtained by using the Univac-1103A.

The axial ratio corresponding to each h can be found from

$$\tanh u_0 = \frac{B}{A} \quad \text{and} \quad e = \sqrt{1 - \tanh^2 u_0}.$$

Note that all the quantities can be obtained from the parameter h. Hence the normalized impedances are only functions of this parameter.

Numerical results obtained by the computer are given in Table I.

The Approximate Impedance

The approximate impedance was obtained using Kihara's approximate field components

$$E_x = 2xy,$$
$$E_y = (2a^2 + b^2)\left(1 - \frac{x^2}{a^2}\right) - y^2$$

major diameter $A=2a$ and minor diameter $B=2b$. These approximate fields do satisfy the boundary conditions.

Now we proceed to calculate the maximum voltage, total axial current and the power flowing in the elliptical waveguide using the approximate fields.

A. The Voltage

The maximum voltage across the waveguide is

$$V = 2\int_0^b E_y dy \Big|_{x=0}$$
$$= 2\int_0^b \left[(2a^2 + b^2)\left(1 - \frac{x^2}{a^2}\right) - y^2\right] dy \Big|_{x=0}$$

performing the integration and the voltage is

$$V = \frac{4}{3}(3a^2b + b^3). \tag{5}$$

B. The Current

The total axial current flowing can be obtained from

$$I = 2\int_0^b \int_0^{x_0} H_{\tan} dx dy \qquad x_0 = a(1 - y^2/b^2)^{1/2},$$

and the expression for the current is

$$I = \frac{4}{3}(2a^3 + ab^2)\frac{1}{Z_0}. \tag{6}$$

C. The Power

The power flowing down the pipe is

$$W = \frac{2}{Z_0}\int_0^b \int_0^{x_0} (E_x^2 + E_y^2) dx dy$$

and the power is

$$W = \frac{\pi}{12 Z_0}(15a^5b + 11a^3b^3 + 2ab^5). \tag{7}$$

The expression for the approximate impedances are

$$Z_{WV} = \frac{32}{3\pi}\frac{r(3 + r^2)}{(5 + 2r^2)} Z_0,$$
$$Z_{WI} = \frac{3\pi}{32}\frac{r(15 + 11r^2 + 2r^4)}{(2 + r^2)^2} Z \tag{8}$$

and

$$Z_{VI} = r\frac{(3 + r^2)}{(2 + r^2)} Z_0$$

where

$$r = \frac{b}{a}.$$

In Figs. 1, 2 and 3 the exact and approximate impedances have been plotted for comparison. The exact cutoff wavelength and Kihara's approximate cutoff wavelength have been shown in Fig. 4.

Kihara obtained the propagation constant for the fundamental mode in elliptic waveguide using his trial fields, and his first approximation is correct to within one per cent:

$$\lambda_c = \pi A\sqrt{\frac{1}{6}\frac{(5 + 2r^2)}{(3 + r^2)}}.$$

Although Chu had already presented the cutoff wavelength for this mode, we present it here once more in a form easier to use. Chu in his paper plotted λ_c/s against eccentricities, where s is the circumference of the elliptical waveguide.

Conclusion

The exact values of the impedances are presented numerically and graphically. Expressions for the impedance of an elliptical waveguide are given. The impedance is obtained within six per cent of the exact value using the approximate results. Comparison of the curves indicates that expressions (8) can give the impedances of elliptical waveguide within three per cent of the exact values by scaling them by a factor of 1.03.

Acknowledgment

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